Introduction

In Chapter Two, we defined a polygon as follows:

**Polygon:** A polygon is a many sided closed figure comprised completely of line segments. Exactly two line segments meet at each endpoint. Each mentioned endpoint is called a vertex and each mentioned segment is called a side of the polygon.

In this chapter we would like to study polygons in detail and discover some of their properties and also study triangles and quadrilaterals as two special classes of polygons.

4.1. Polygons

Polygons are all round us. They can be used to convey information: Knowing the color and shape will allow you to recognize what information will be presented before you can actually read the sign while you are driving. The nature is full of polygons that convey its sense of symmetry and design. We study polygons because they occur in our world.

A **convex** polygon has the property that the line segment \(PQ\) generated from any two arbitrary points \(P\) and \(Q\) inside the polygon lies entirely inside the polygon. (Figure 4.1.2.a)

![Figure 4.1.1](image1.png)

![Figure 4.1.2](image2.png)
In the case where we can find two points $A$ and $B$ inside of a polygon such that the line segment $AB$ does not lie entirely inside the polygon, the polygon is called **nonconvex**. (Figure 4.1.2.b)

From now on, by polygon we mean a convex polygon unless otherwise stated.

A **diagonal** of a polygon is a line segment that joins two non-adjacent vertices.

In order to refer to a polygon with respect to its number of sides $n$, we use the word **$n$-gon**. Therefore a triangle is a 3-gon and we do not have a 2-gon in Euclidean geometry (We have this case in spherical geometry!).

**Theorem 4.1:** *Any $n$-gon can be divided into $n-2$ triangles.*

To convince someone to accept this assertion, we should show the case for a general $n$. But $n$ is a variable and not a fixed value. What we can do is to study several specific $n$-gon cases such as heptagon (7-gon) and octagon (8-gon) and then try to generalize the idea for all other cases.

In general, we can say if we have an $n$-gon, then we have $n$ vertices that we can label them from $A_1$ to $A_n$. The first two sides of $A_1A_2$ and $A_2A_3$ with the diagonal $A_1A_3$ identify our first triangle. (We used two sides to build one triangle—we lost one extra side compared to the constructed triangles). After that, each side, with the help of two adjacent diagonals identify one new triangle (Here, we use one side to build one triangle). But the last triangle will employ the last two sides of $A_1A_n$ and $A_nA_{n-1}$ to be constructed (two sides for one triangle—we lost another extra side here to gain a triangle). This shows the number of constructed triangles are two less than the number of sides!

The following is an important theorem that gives us the ability to calculate the sum of angle measures of any polygon:

**Theorem 4.2:** *The sum of angle measures of an $n$-gon is $(n-2)180^\circ$.*

To show that this theorem is valid we notice that from the previous theorem we can divide the $n$-gon to $n-2$ triangles. In the next section we will show that the sum of angle measures of each triangle is $180^\circ$. Therefore, the sum of angle measures of an $n$-gon is $(n-2)180^\circ$.

The following table summarizes what we have studied related to the past two theorems.
### Table 4.1.1

<table>
<thead>
<tr>
<th>Number of Sides</th>
<th>Number of Diagonals Emanated from a Vertex</th>
<th>Number of Constructed Triangles</th>
<th>Sum of Angle Measures</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>180°</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2</td>
<td>2 × 180° = 360°</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>3</td>
<td>3 × 180° = 540°</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( n )</td>
<td>( n - 3 )</td>
<td>( n - 2 )</td>
<td>( (n - 2) \times 180° = (n - 2)180° )</td>
</tr>
</tbody>
</table>

### Regular Polygons

A regular polygon is a polygon with all sides congruent and all angles congruent. We have already been introduced to regular polygons such as equilateral triangles, squares, regular pentagon, and regular hexagon.

We notice since all angles of a regular \( n \)-gon are congruent and since its sum of angle measures is \((n - 2)180°\), we are able to find the degrees measure of each angle as follows:

\[
\text{Number of Degrees in one Angle} = \frac{(n - 2)180°}{n}
\]

**Note:** In some literatures the angle of a polygon is called “interior angle”.

Using the above formula, Table 4.1.2 presents information about some regular polygons.

### Table 4.1.2

<table>
<thead>
<tr>
<th>Regular Polygon</th>
<th>Number of Sides</th>
<th>Number of Degrees in One Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral Triangle</td>
<td>3</td>
<td>60°</td>
</tr>
<tr>
<td>Square</td>
<td>4</td>
<td>90°</td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td>108°</td>
</tr>
<tr>
<td>Hexagon</td>
<td>6</td>
<td>120°</td>
</tr>
<tr>
<td>Heptagon</td>
<td>7</td>
<td>128 4/7°</td>
</tr>
<tr>
<td>Octagon</td>
<td>8</td>
<td>135°</td>
</tr>
<tr>
<td>Nonagon</td>
<td>9</td>
<td>140°</td>
</tr>
<tr>
<td>Decagon</td>
<td>10</td>
<td>144°</td>
</tr>
</tbody>
</table>

The center of a regular polygon is a point inside of the polygon that is equidistance from all the vertices. An angle with a vertex at the center of a regular polygon and with its sides passing through two adjacent vertices of the polygon is called a **central angle**. We note that a regular \( n \)-gon contains \( n \) congruent central angles. The sum of angle measures of all central angles is 360°. Therefore we are able to establish the following formula:

\[
\text{Number of Degrees in one Central Angle} = \frac{360°}{n}
\]
Rotational Symmetry and Regular Polygons

Suppose that you trace the design in Figure 4.1.6 by a pencil and then rotate the tracing design, which is on the top of the original one, about its center, $O$. You will notice that in less than one full turn the tracing design and the original figure will coincide. In fact, this will happen exactly three times. Such a figure is said to have rotational symmetry. The point $O$ is called the center of rotation. Since we could coincide the tracing and original designs three times in one full turn, we say that the figure has three-fold (3-fold) rotational symmetry.

All regular polygons have rotational symmetry. The center of rotation is the center of the polygon. A regular $n$-gon has $n$-fold rotational symmetry.
Reflective Symmetry and Regular Polygons

Consider that you fold this page along the vertical dashed line in Figure 4.1.8. Then you will notice that one half of the figure will fit exactly over the other half. If you place a mirror along the dashed line, you would observe that the mirror image of one half of the design is exactly the same as the other half. We say that such a design has reflective symmetry. The dashed line \( l \) is said to be the line of reflection. Since the design has only one line of reflection we say that it has one-fold (1-fold) reflective symmetry.

All regular polygons have reflective symmetry. The lines of reflections are concurrent at the center of the polygon. A regular \( n \)-gon has \( n \)-fold reflective symmetry.

Exercise Set 4.1

1. Find the sum of angles of a heptagon, nonagon, 52-gon, and 100-gon.

2. Use only a compass and straightedge to construct a regular 3-gon, 6-gon, and 12-gon (dodecagon). Repeat the problem by using the Geometer’s Sketchpad.

3. Find the angle and the central angle of the following regular polygons:
4. Find the measures of angles and central angles of a regular

(a) pentagon, (b) heptagon, (c) nonagon, (d) dodecagon, (e) and 60-gon.

5. For each of the designs in Figure 4.1.11 do as follows:

(a) If it has rotational symmetry, find its center of rotation and the number of its rotational symmetry folds.

(b) If it has reflective symmetry, find its lines of reflection and the number of its reflective symmetry folds.

![Figure 4.1.11](image)

6. Recall the theorem about the constructible regular polygons mentioned in Chapter Three that was proved by Gauss. List all regular polygons with fewer than 50 sides that can be constructed with compass and straightedge.

7. Using Sketchpad we are able to construct shapes with rotational symmetries. Take a look at Figure 4.1.12.a. It is a nonconvex polygon that has a right angle. Generate this polygon in the Geometer’s Sketchpad (except for the right angle you don’t need to be accurate to produce it exactly). Now select $O$ and then in “Transform” menu click on “Mark Center”. Now select the entire figure and go back to “Transform” but this time choose “Rotate”. A screen will appear and asks you about the angle of rotation. Choose 90° and then click OK. You will see that if you repeat this procedure two more times you have Figure 4.1.12.b, which has 4-fold rotational symmetry (and not reflective symmetry).
Now construct a nonconvex polygon with a certain angle to create the following:

(a) A design with 8-fold rotational symmetry that does not have reflective symmetry.
(b) A design with 6-fold rotational symmetry that does not have reflective symmetry.
(c) A design with 5-fold rotational symmetry that does not have reflective symmetry.

8. Using Sketchpad we also are able to construct shapes with reflective symmetries. Consider Figure 4.1.13.a. It is a nonconvex polygon that has a right angle. Generate this polygon in the Geometer’s Sketchpad. Now select the line segment $AB$ and then in “Transform” menu click on “Mark Mirror”. Then select your entire figure and go back to “Transform” and this time choose “Reflect”. You will see that if you repeat this procedure by selecting $BC$ and its mirror image as your new “Mirror” you will obtain Figure 4.1.13.b, which has 2-fold reflective symmetry.